

# The Bargmann representation for the quantum mechanics on a sphere

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The Bargmann representation is constructed corresponding to the coherent states for a particle on a sphere introduced in: K. Kowalski and J. Rembieliński, J. Phys. A: Math. Gen. **33**, 6035 (2000). The connection is discussed between the introduced formalism and the standard approach based on the Hilbert space of square integrable functions on a sphere  $S^2$ .

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## I. INTRODUCTION

In our recent paper [1] the coherent states for a particle on a sphere have been introduced. As with the standard coherent states [2] those states are labelled by points of the classical phase space i.e. the cotangent bundle  $T^*S^2$ . It is worthwhile to recall that the celebrated spin coherent states [3] are not related to the phase space for a particle on the sphere  $S^2$ . One of the most characteristic properties of coherent states is the existence of the Bargmann representation. Such representation is of importance not only from the mathematical point of view. An example of applications are the Husimi functions i.e. the elements of the Bargmann space, in the theory of quantum chaos. In this work we introduce the Bargmann representation referring to the coherent states for a particle on a sphere mentioned above. It should be noted that, in opposition to the case of the standard coherent states, the construction of such Bargmann representation is a highly nontrivial problem. The paper is organized as follows. In section II we recall the basic properties of the coherent states for a particle on a sphere. Sections III–V are devoted to the construction of the Bargmann representation. In section VI we discuss the connection of the introduced Bargmann representation and the standard coordinate representation for the quantum mechanics on a sphere.

## II. COHERENT STATES FOR A PARTICLE ON A SPHERE

Our purpose in this section is to recall the basic properties of the coherent states for a particle on a sphere introduced in [1]. Those states are related to the  $e(3)$  algebra of the form

$$[J_i, J_j] = i\varepsilon_{ijk} J_k, \quad [J_i, X_j] = i\varepsilon_{ijk} X_k, \quad [X_i, X_j] = 0, \quad i, j, k = 1, 2, 3. \quad (2.1)$$

The Casimir operators are given in a unitary irreducible representation by

$$\mathbf{X}^2 = r^2, \quad \mathbf{J} \cdot \mathbf{X} = \lambda, \quad (2.2)$$

where dot designates the scalar product. In [1] we restricted to the special case  $\lambda = 0$ , so

$$\mathbf{J} \cdot \mathbf{X} = 0. \quad (2.3)$$

The irreducible representation of (2.1) under the choice (2.3) is spanned by the common eigenvectors  $|j, m; r\rangle$  of the operators  $\mathbf{J}^2$ ,  $\mathbf{X}^2$  and  $\mathbf{J} \cdot \mathbf{X}$ . We have

$$\mathbf{J}^2 |j, m; r\rangle = j(j+1) |j, m; r\rangle, \quad J_3 |j, m; r\rangle = m |j, m; r\rangle, \quad (2.4a)$$

$$\mathbf{X}^2 |j, m; r\rangle = r^2 |j, m; r\rangle, \quad (\mathbf{J} \cdot \mathbf{X} / r) |j, m; r\rangle = 0, \quad (2.4b)$$

where  $-j \leq m \leq j$ . The operators  $J_{\pm} = J_1 \pm iJ_2$ ,  $X_{\pm} = X_1 \pm iX_2$  and  $X_3$  act on the vectors  $|j, m; r\rangle$  as follows

$$J_{\pm}|j, m; r\rangle = \sqrt{(j \mp m)(j \pm m + 1)}|j, m \pm 1; r\rangle, \quad (2.5a)$$

$$\begin{aligned} X_{\pm}|j, m; r\rangle = & \mp \frac{r\sqrt{(j \pm m + 1)(j \pm m + 2)}}{\sqrt{(2j + 1)(2j + 3)}}|j + 1, m \pm 1; r\rangle \\ & \pm \frac{r\sqrt{(j \mp m - 1)(j \mp m)}}{\sqrt{(2j - 1)(2j + 1)}}|j - 1, m \pm 1; r\rangle, \end{aligned} \quad (2.5b)$$

$$\begin{aligned} X_3|j, m; r\rangle = & \frac{r\sqrt{(j - m + 1)(j + m + 1)}}{\sqrt{(2j + 1)(2j + 3)}}|j + 1, m; r\rangle \\ & + \frac{r\sqrt{(j - m)(j + m)}}{\sqrt{(2j - 1)(2j + 1)}}|j - 1, m; r\rangle. \end{aligned} \quad (2.5c)$$

The orthogonality and completeness conditions satisfied by the vectors  $|j, m; r\rangle$  can be written as

$$\langle j, m; r | j', m'; r \rangle = \delta_{jj'} \delta_{mm'}, \quad (2.6)$$

$$\sum_{j=0}^{\infty} \sum_{m=-j}^j |j, m; r\rangle \langle j, m; r| = I, \quad (2.7)$$

where  $I$  is the identity operator.

We are now in a position to introduce the coherent states for a particle on a sphere. Namely, these states are defined as the solution of the eigenvalue equation of the form

$$\mathbf{Z}|\mathbf{z}\rangle = \mathbf{z}|\mathbf{z}\rangle, \quad (2.8)$$

where  $\mathbf{Z}$  is given by

$$\begin{aligned} \mathbf{Z} = & \left( \frac{e^{\frac{1}{2}}}{\sqrt{1 + 4\mathbf{J}^2}} \sinh \frac{1}{2} \sqrt{1 + 4\mathbf{J}^2} + e^{\frac{1}{2}} \cosh \frac{1}{2} \sqrt{1 + 4\mathbf{J}^2} \right) \frac{\mathbf{X}}{r} \\ & + i \left( \frac{2e^{\frac{1}{2}}}{\sqrt{1 + 4\mathbf{J}^2}} \sinh \frac{1}{2} \sqrt{1 + 4\mathbf{J}^2} \right) \mathbf{J} \times \frac{\mathbf{X}}{r}, \end{aligned} \quad (2.9)$$

where the cross designates the vector product. The operator  $\mathbf{Z}$  and  $\mathbf{z} \in \mathbf{C}^3$  obey

$$\mathbf{Z}^2 = 1, \quad \mathbf{z}^2 = 1. \quad (2.10)$$

We also write down the following matrix representation of the operator  $\mathbf{Z}$  which is crucial for the algebraic analysis of the problem:

$$e^{-i(\boldsymbol{\sigma} \cdot \mathbf{J} + 1)} \boldsymbol{\sigma} \cdot \mathbf{X} = \boldsymbol{\sigma} \cdot \mathbf{Z}, \quad (2.11)$$

where  $\sigma_i$ ,  $i = 1, 2, 3$ , are the Pauli matrices.

As with the standard coherent states we can generate the coherent states from the “fiducial vector”  $|\mathbf{n}_3\rangle$  such that

$$\mathbf{Z}|\mathbf{n}_3\rangle = \mathbf{n}_3|\mathbf{n}_3\rangle, \quad (2.12)$$

where  $\mathbf{n}_3 = (0, 0, 1)$ , and

$$|\mathbf{n}_3\rangle = \sum_{j=0}^{\infty} e^{-\frac{1}{2}j(j+1)} \sqrt{2j+1} |j, 0; r\rangle. \quad (2.13)$$

Namely, the coherent states are given by

$$|\mathbf{z}\rangle = \exp \left[ \frac{\operatorname{arccosh} z_3}{\sqrt{1-z_3^2}} (\mathbf{z} \times \mathbf{n}_3) \cdot \mathbf{J} \right] |\mathbf{n}_3\rangle. \quad (2.14)$$

The projection of the coherent states (2.14) on the discrete basis vectors  $|j, m; r\rangle$  is

$$\langle j, m; r | \mathbf{z} \rangle = e^{-\frac{1}{2}j(j+1)} \sqrt{2j+1} \frac{(2|m|)!}{|m|!} \sqrt{\frac{(j-|m|)!}{(j+|m|)!}} \left( \frac{-\varepsilon(m)z_1 + iz_2}{2} \right)^{|m|} C_{j-|m|}^{|m|+\frac{1}{2}}(z_3), \quad (2.15)$$

where  $\varepsilon(m)$  is the sign of  $m$ , and  $C_n^\alpha(x)$  are the Gegenbauer polynomials expressed with the help of the hypergeometric function  ${}_2F_1(a, b, c; z)$  by

$$C_n^\alpha(x) = \frac{\Gamma(n+2\alpha)}{\Gamma(n+1)\Gamma(2\alpha)} {}_2F_1(-n, n+2\alpha, \alpha + \frac{1}{2}; \frac{1}{2}(1-x)). \quad (2.16)$$

As mentioned earlier the coherent states are labelled by points of the classical phase space  $T^*S^2$ . The most natural complex parametrization of the phase space discussed in [1] is of the form

$$\mathbf{z} = \cosh |\mathbf{l}| \frac{\mathbf{x}}{r} + i \frac{\sinh |\mathbf{l}|}{|\mathbf{l}|} \mathbf{l} \times \frac{\mathbf{x}}{r}, \quad (2.17)$$

where the vectors  $\mathbf{l}, \mathbf{x} \in \mathbf{R}^3$ , fulfil  $\mathbf{x}^2 = r^2$  and  $\mathbf{l} \cdot \mathbf{x} = 0$ , that is  $\mathbf{l}$  is the classical angular momentum and  $\mathbf{x}$  is the radius vector of a particle on a sphere. Clearly, the vector  $\mathbf{z}$  satisfies the second equation of (2.10).

### III. SCALAR PRODUCT

In this section we identify the Bargmann space of analytic functions corresponding to the coherent states for a particle on a sphere described above. We now restrict, without loss of generality, to the case with the unit sphere. On introducing the spherical coordinates  $\mathbf{x} = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)$ , and parametrizing the tangent vector  $\mathbf{l}$  by its norm  $|\mathbf{l}| \equiv l$  and the angle  $\alpha$  between  $\mathbf{l}$  and the meridian passing through the point with the radius vector  $\mathbf{x}$ , we obtain from (2.17) the following natural coordinates of the phase space compatible with the constraints:

$$\begin{aligned} z_1 &= \cosh l \sin \theta \cos \varphi + i \sinh l (\sin \alpha \cos \varphi \cos \theta - \cos \alpha \sin \varphi), \\ z_2 &= \cosh l \sin \theta \sin \varphi + i \sinh l (\sin \alpha \sin \varphi \cos \theta + \cos \alpha \cos \varphi), \\ z_3 &= \cosh l \cos \theta - i \sinh l \sin \alpha \sin \theta. \end{aligned} \quad (3.1)$$

Taking into account the fact that  $\mathbf{z}$  transforms as the vector we find that the Bargmann space should be specified by

$$\langle \phi | \psi \rangle = \frac{1}{8\pi^2} \int_0^{2\pi} d\varphi \int_0^\pi \sin \theta d\theta \int_0^{2\pi} d\alpha \int_0^\infty dl h(l) (\phi(\mathbf{z}^*(\theta, \varphi, \alpha, l)))^* \psi(\mathbf{z}^*(\theta, \varphi, \alpha, l)), \quad (3.2)$$

where  $\phi(\mathbf{z}^*) = \langle \mathbf{z} | \phi \rangle$ ,  $\mathbf{z}^* = (z_1^*, z_2^*, z_3^*)$ ,  $h(l)$  is an unknown density and  $\mathbf{z}(\theta, \varphi, \alpha, l)$  is expressed by (3.1). Clearly, the corresponding resolution of the identity can be written in the form

$$\frac{1}{8\pi^2} \int_0^{2\pi} d\varphi \int_0^\pi \sin \theta d\theta \int_0^{2\pi} d\alpha \int_0^\infty dl h(l) |\mathbf{z}(\theta, \varphi, \alpha, l)\rangle \langle \mathbf{z}(\theta, \varphi, \alpha, l)| = I. \quad (3.3)$$

In order to fix  $h(l)$  consider the basis of the Bargmann space with the scalar product (3.2)

$$e_{jm}(\mathbf{z}(\theta, \varphi, \alpha, l)) = \langle j, m | \mathbf{z} \rangle, \quad (3.4)$$

where  $|j, m\rangle \equiv |j, m; 1\rangle$  and  $\langle j, m; r|\mathbf{z}\rangle$  is given by (2.15) and (3.1). Using (3.2) and (3.4) as well as some guessing work we get

$$\begin{aligned} & \langle j, m|j', m'\rangle \\ &= \frac{1}{8\pi^2} \int_0^{2\pi} d\varphi \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\alpha \int_0^\infty dl h(l) (e_{jm}(\mathbf{z}^*(\theta, \varphi, \alpha, l))^* e_{j'm'}(\mathbf{z}^*(\theta, \varphi, \alpha, l))) \\ &= \delta_{jj'} \delta_{mm'} e^{-j(j+1)} \int_0^\infty dl h(l) P_j(\cosh 2l), \end{aligned} \quad (3.5)$$

where  $P_n(z)$  are the Legendre polynomials such that

$$P_n(z) = \frac{1}{2^n n!} \frac{d^n}{dz^n} (z^2 - 1)^n. \quad (3.6)$$

Thus the normalization condition for the orthonormal basis  $\{|j, m\rangle\}$  leads to the following equation on the density  $h(l)$ :

$$\int_0^\infty dl h(l) P_j(\cosh 2l) = e^{j(j+1)}. \quad (3.7)$$

We remark that the problem of the solution of this equation is highly nontrivial (see acknowledgements) and it is related to the so called problem of moments [4]. We now recall that the Legendre polynomials satisfy the differential equation

$$\left( (z^2 - 1) \frac{d^2}{dz^2} + 2z \frac{d}{dz} \right) P_n(z) = n(n+1) P_n(z). \quad (3.8)$$

From (3.8) it follows easily that

$$\frac{1}{\sinh \rho} \frac{d}{d\rho} \sinh \rho \frac{d}{d\rho} P_n(\cosh \rho) = n(n+1) P_n(\cosh \rho). \quad (3.9)$$

We remark that the operator from the left hand side of (3.9) is simply the Laplacian for the two-dimensional hyperbolic space. Consider the heat kernel at the origin in hyperbolic space [5], given by

$$k_{H^2}(\rho, t) = 2^{\frac{1}{2}} (4\pi t)^{-\frac{3}{2}} e^{-\frac{\rho^2}{4t}} \int_0^\infty \frac{s e^{-\frac{s^2}{4t}}}{(\cosh s - \cosh \rho)^{\frac{1}{2}}} ds. \quad (3.10)$$

This heat kernel obeys the equation

$$\frac{\partial k_{H^2}}{\partial t} = \frac{1}{\sinh \rho} \frac{d}{d\rho} \sinh \rho \frac{d}{d\rho} k_{H^2}(\rho, t), \quad (3.11)$$

subject to the initial condition

$$2\pi \lim_{t \rightarrow 0} \int_0^\infty k_{H^2}(\rho, t) f(\rho) \sinh \rho d\rho = f(0), \quad (3.12)$$

where  $f$  is an arbitrary continuous function with at most exponential growth at infinity. Putting  $f(\rho) = P_n(\cosh \rho)$ , and making use of (3.11), (3.12) and the fact that  $k_{H^2}(\rho, t)$  and  $\frac{d}{d\rho} k_{H^2}(\rho, t)$  decay faster-than-exponentially, we get

$$2\pi \int_0^\infty k_{H^2}(\rho, t) P_n(\cosh \rho) \sinh \rho d\rho = e^{tn(n+1)}. \quad (3.13)$$

Hence, setting in (3.13)  $\rho = 2l$  and  $t = 1$ , we finally find that the desired density  $h(l)$  satisfying (3.7) is

$$h(l) = 4\pi k_{H^2}(2l, 1) \sinh 2l \quad (3.14a)$$

$$= \frac{e^{-\frac{1}{4}} \sinh 2l}{\sqrt{2\pi}} \int_{2l}^\infty \frac{se^{-\frac{s^2}{4}}}{(\cosh s - \cosh 2l)^{\frac{1}{2}}} ds. \quad (3.14b)$$

We have thus identified the Bargmann space for the quantum mechanics on a sphere specified by (3.2) and (3.14). Taking into account (3.1), (3.14a) and the relation which is an immediate consequence of (2.17) such that

$$\mathbf{z} \cdot \mathbf{z}^* = |\mathbf{z}|^2 = \cosh 2l, \quad (3.15)$$

the following form can be derived of the scalar product (3.2) written with the help of the complex variables  $\mathbf{z}$  (3.1) analogous to the usual Bargmann representation [6] for the standard coherent states:

$$\langle \phi | \psi \rangle = \int_{\mathbf{z}^2=1} d\mu(\mathbf{z}) (\phi(\mathbf{z}^*))^* \psi(\mathbf{z}^*), \quad (3.16)$$

where

$$d\mu(\mathbf{z}) = \frac{1}{4\pi} k_{H^2}(\operatorname{arccosh}(\mathbf{z} \cdot \mathbf{z}^*), 1) dz_1 dz_2 dz_3 dz_1^* dz_2^* dz_3^*, \quad (3.17)$$

and  $\phi(\mathbf{z}^*) = \langle \mathbf{z} | \phi \rangle$ . Evidently, the completeness of the coherent states can be written with the help of the measure  $d\mu(\mathbf{z})$  as

$$\int_{\mathbf{z}^2=1} d\mu(\mathbf{z}) |\mathbf{z}\rangle \langle \mathbf{z}| = I. \quad (3.18)$$

#### IV. REPRODUCING KERNEL

As is well-known the existence of the reproducing kernel is one of the most characteristic properties of coherent states. In view of (3.18) the reproducing property can be written in the form

$$\phi(\mathbf{w}^*) = \int_{\mathbf{z}^2=1} d\mu(\mathbf{z}) \mathcal{K}(\mathbf{w}^*, \mathbf{z}) \phi(\mathbf{z}^*), \quad (4.1)$$

where  $\phi(\mathbf{w}^*) = \langle \mathbf{w} | \phi \rangle$ , and

$$\mathcal{K}(\mathbf{w}^*, \mathbf{z}) = \langle \mathbf{w} | \mathbf{z} \rangle. \quad (4.2)$$

It should be noted that the reproducing kernel  $\mathcal{K}(\mathbf{w}^*, \mathbf{z})$  is the complex conjugate of the analytic function

$$\phi_{\mathbf{w}}(\mathbf{z}^*) = \langle \mathbf{z} | \mathbf{w} \rangle \quad (4.3)$$

representing the abstract coherent state  $|\mathbf{w}\rangle$  also called its symbol. The formula on the overlap  $\langle \mathbf{z} | \mathbf{w} \rangle$  can be obtained from (2.14), (2.13) and (2.15). Namely, we have

$$\langle \mathbf{z} | \mathbf{w} \rangle = \sum_{j=0}^{\infty} e^{-j(j+1)} (2j+1) P_j(\mathbf{z}^* \cdot \mathbf{w}), \quad (4.4)$$

where  $P_j(z)$  are the Legendre polynomials given by (3.6).

## V. ACTION OF OPERATORS

We now discuss the action of operators in the Bargmann representation. We first observe that an immediate consequence of (2.8) is the following formula on the action of operators  $\mathbf{Z}^\dagger$ :

$$\mathbf{Z}^\dagger \phi(\mathbf{z}^*) = \mathbf{z}^* \phi(\mathbf{z}^*), \quad (5.1)$$

where  $\phi(\mathbf{z}^*) = \langle \mathbf{z} | \phi \rangle$  and we recall that  $\mathbf{z}^2 = 1$ . Now consider the action of the operator  $\mathbf{J}^2$ . By (2.4a) the action of the operator  $\mathbf{J}^2$  on the basis  $e_{jm}(\mathbf{z}^*) = \langle \mathbf{z} | j, m \rangle$  of the Bargmann space is the following one:

$$\mathbf{J}^2 e_{jm}(\mathbf{z}^*) = j(j+1) e_{jm}(\mathbf{z}^*). \quad (5.2)$$

Using (2.15), the differential equation satisfied by the Gegenbauer polynomials of the form

$$\left( (z^2 - 1) \frac{d^2}{dz^2} + (2\lambda + 1) \frac{d}{dz} - n(2\lambda + n) \right) C_n^\lambda(z) = 0, \quad (5.3)$$

and (5.2) we find that the operator  $\mathbf{J}^2$  acts in the representation (3.16) as follows

$$\mathbf{J}^2 \phi(\mathbf{z}^*) = - \left( \mathbf{z}^* \times \frac{\partial}{\partial \mathbf{z}^*} \right)^2 \phi(\mathbf{z}^*). \quad (5.4)$$

Taking into account (5.4) and (5.1) we obtain

$$\mathbf{J} \phi(\mathbf{z}^*) = -i \left( \mathbf{z}^* \times \frac{\partial}{\partial \mathbf{z}^*} \right) \phi(\mathbf{z}^*). \quad (5.5)$$

The relation (5.5) can be easily checked on the basis  $e_{jm}(\mathbf{z}^*)$  with the help of (2.15), (2.4a) and (2.5a). Further, using (2.5c), (2.15) and elementary properties of the Gegenbauer polynomials we get

$$X_3 \phi(\mathbf{z}^*) = e^{-\frac{1}{2} \mathbf{J}^2} z_3^* e^{\frac{1}{2} \mathbf{J}^2} \phi(\mathbf{z}^*), \quad (5.6)$$

where the action of  $\mathbf{J}^2$  is given by (5.4). The action of the remaining coordinates of the position operator  $\mathbf{X}$  can be obtained by means of the following identity describing the complex rotation of  $\mathbf{X}$ :

$$e^{\mathbf{w} \cdot \mathbf{J}} \mathbf{X} e^{-\mathbf{w} \cdot \mathbf{J}} = \cosh \sqrt{\mathbf{w}^2} \mathbf{X} - i \frac{\sinh \sqrt{\mathbf{w}^2}}{\sqrt{\mathbf{w}^2}} \mathbf{w} \times \mathbf{X} + \frac{1 - \cosh \sqrt{\mathbf{w}^2}}{\mathbf{w}^2} \mathbf{w} (\mathbf{w} \cdot \mathbf{X}). \quad (5.7)$$

Namely, we have

$$X_1 \phi(\mathbf{z}^*) = -\frac{i}{\sinh 1} \left( e^{J_2 - \frac{1}{2} \mathbf{J}^2} z_3^* e^{\frac{1}{2} \mathbf{J}^2 - J_2} - \cosh 1 e^{-\frac{1}{2} \mathbf{J}^2} z_3^* e^{\frac{1}{2} \mathbf{J}^2} \right) \phi(\mathbf{z}^*), \quad (5.8a)$$

$$X_2 \phi(\mathbf{z}^*) = \frac{i}{\sinh 1} \left( e^{J_1 - \frac{1}{2} \mathbf{J}^2} z_3^* e^{\frac{1}{2} \mathbf{J}^2 - J_1} - \cosh 1 e^{-\frac{1}{2} \mathbf{J}^2} z_3^* e^{\frac{1}{2} \mathbf{J}^2} \right) \phi(\mathbf{z}^*), \quad (5.8b)$$

where the action of the operators  $J_i$ ,  $i = 1, 2$ , and  $\mathbf{J}^2$  is given by (5.5) and (5.4), respectively. Finally, taking into account the identity

$$\mathbf{Z} = e^{-\frac{1}{2} \mathbf{J}^2} \mathbf{X} e^{\frac{1}{2} \mathbf{J}^2}, \quad (5.9)$$

which is a straightforward consequence of (2.11) and the commutation relation

$$[\mathbf{J}^2, \boldsymbol{\sigma} \cdot \mathbf{X}] = -2(\boldsymbol{\sigma} \cdot \mathbf{J} + 1) \boldsymbol{\sigma} \cdot \mathbf{X}, \quad (5.10)$$

following directly from (2.1) and (2.3), we obtain the action of the operator  $\mathbf{Z}$ . It follows that

$$Z_1 \phi(\mathbf{z}^*) = -\frac{i}{\sinh 1} \left( e^{J_2 - \mathbf{J}^2} z_3^* e^{\mathbf{J}^2 - J_2} - \cosh 1 e^{-\mathbf{J}^2} z_3^* e^{\mathbf{J}^2} \right) \phi(\mathbf{z}^*), \quad (5.11a)$$

$$Z_2 \phi(\mathbf{z}^*) = \frac{i}{\sinh 1} \left( e^{J_1 - \mathbf{J}^2} z_3^* e^{\mathbf{J}^2 - J_1} - \cosh 1 e^{-\mathbf{J}^2} z_3^* e^{\mathbf{J}^2} \right) \phi(\mathbf{z}^*), \quad (5.11b)$$

$$Z_3 \phi(\mathbf{z}^*) = e^{-\mathbf{J}^2} z_3^* e^{\mathbf{J}^2} \phi(\mathbf{z}^*). \quad (5.11c)$$

## VI. THE BARGMANN REPRESENTATION AND THE COORDINATE REPRESENTATION

In this section we discuss the relationship between the introduced Bargmann representation and the standard coordinate representation for the quantum mechanics on a sphere. We begin with recalling the basic facts about the coordinate representation. Consider the position operators  $\mathbf{X}$  for a particle on a sphere satisfying the  $e(3)$  algebra (2.1). Recall that we restrict to the irreducible representations which fulfil (2.3) and  $\mathbf{X}^2 = 1$ . The coordinate representation is spanned by the common eigenvectors  $|\mathbf{x}\rangle$  of the position operators such that

$$\mathbf{X}|\mathbf{x}\rangle = \mathbf{x}|\mathbf{x}\rangle, \quad (6.1)$$

where  $\mathbf{x}^2 = 1$ . The resolution of the identity is of the form

$$\int_{\mathbf{x}^2=1} d\nu(\mathbf{x}) |\mathbf{x}\rangle \langle \mathbf{x}| = I, \quad (6.2)$$

where  $d\nu(\mathbf{x}) = d\nu(\theta, \varphi) = \sin\theta d\varphi d\theta$ , accordingly to the natural i.e. spherical coordinates  $\mathbf{x} = (\sin\theta \cos\varphi, \sin\theta \sin\varphi, \cos\theta)$  compatible with the constraint  $\mathbf{x}^2 = 1$ . The completeness gives rise to a functional representation of vectors such that

$$\langle \phi | \psi \rangle = \int_{\mathbf{x}^2=1} d\nu(\mathbf{x}) \phi^*(\mathbf{x}) \psi(\mathbf{x}), \quad (6.3)$$

where  $\phi(\mathbf{x}) = \langle \mathbf{x} | \phi \rangle$ . Clearly, we can write the completeness condition (6.2) and the scalar product (6.3) as

$$\int_0^{2\pi} d\varphi \int_0^\pi \sin\theta d\theta |\theta, \varphi\rangle \langle \theta, \varphi| = I, \quad (6.4)$$

where  $|\theta, \varphi\rangle \equiv |\mathbf{x}\rangle$ , and

$$\langle \phi | \psi \rangle = \int_0^{2\pi} d\varphi \int_0^\pi \sin\theta d\theta \phi^*(\theta, \varphi) \psi(\theta, \varphi), \quad (6.5)$$

where  $\phi(\theta, \varphi) = \langle \theta, \varphi | \phi \rangle$ , respectively. The passage from the coordinate representation to the angular momentum representation generated by the vectors  $|j, m\rangle$  satisfying (2.4) with  $r = 1$  is given by

$$\langle \theta, \varphi | j, m \rangle = Y_{jm}(\theta, \varphi) = (-1)^{\frac{m-|m|}{2}} \sqrt{\frac{(2j+1)(j-|m|)!}{4\pi(j+|m|)!}} P_j^{|m|}(\cos\theta) e^{im\varphi}, \quad (6.6)$$

where  $Y_{jm}(\theta, \varphi)$  are the spherical harmonics and  $P_n^m(z)$  are the associated Legendre polynomials which can be defined by

$$P_n^m(z) = (-1)^m (1-z^2)^{\frac{m}{2}} \frac{d^m}{dz^m} P_n(z), \quad (6.7)$$

where  $P_n(z)$  are the Legendre polynomials given by (3.6). Of course,  $Y_{jm}(\theta, \varphi)$  form the orthonormal basis of the Hilbert space of the square integrable functions on a sphere  $S^2$  specified by the scalar product (6.5). Taking into account (6.6) and the identity

$$C_{n-m}^{m+\frac{1}{2}}(z) = (-1)^m \frac{(1-z^2)^{-\frac{m}{2}} m! 2^m}{(2m)!} P_n^m(z), \quad (6.8)$$

where  $m+1$  is natural, we find that the kernel (6.6) can be written in the form analogous to (2.15) such that

$$\langle \mathbf{x}|j, m\rangle = \sqrt{\frac{2j+1}{4\pi}} \frac{(2|m|)!}{|m|!} \sqrt{\frac{(j-|m|)!}{(j+|m|)!}} \left( \frac{-\varepsilon(m)x_1 - ix_2}{2} \right)^{|m|} C_{j-|m|}^{|m|+\frac{1}{2}}(x_3). \quad (6.9)$$

Now, let  $|\mathbf{x}, \mathbf{l}\rangle$  designate the coherent state  $|\mathbf{z}\rangle$  in accordance with the parametrization of the phase space given by (2.17). Eqs. (5.9), (2.15) and (6.9) taken together yield

$$|\mathbf{x}, \mathbf{0}\rangle = \sqrt{4\pi} e^{-\frac{1}{2}\mathbf{J}^2} |\mathbf{x}\rangle. \quad (6.10)$$

Using (2.14), (6.10) and (2.13) and proceeding as with (4.4) we find that the passage from the coordinate representation to the coherent states representation is described by the matrix element

$$\langle \mathbf{x}|\mathbf{z}\rangle = \frac{1}{\sqrt{4\pi}} \sum_{j=0}^{\infty} e^{-\frac{1}{2}j(j+1)} (2j+1) P_j(\mathbf{x}\cdot\mathbf{z}). \quad (6.11)$$

Evidently, (6.11) defines a unitary map  $U : \phi \rightarrow \tilde{\phi}$  from the standard Hilbert space of square integrable functions on the sphere  $S^2$  with the scalar product (6.3) onto the Bargmann space of analytic functions specified by the scalar product (3.16), of the form

$$(U\phi)(\mathbf{z}^*) = \int_{\mathbf{x}^2=1} d\nu(\mathbf{x}) k(\mathbf{x}\cdot\mathbf{z}^*) \phi(\mathbf{x}), \quad (6.12)$$

where  $k(\mathbf{x}\cdot\mathbf{z}) = \langle \mathbf{x}|\mathbf{z}\rangle$ . The inverse operator  $U^{-1}$  is given by

$$(U^{-1}\tilde{\phi})(\mathbf{x}) = \int_{\mathbf{z}^2=1} d\mu(\mathbf{z}) k(\mathbf{x}\cdot\mathbf{z}) \tilde{\phi}(\mathbf{z}^*). \quad (6.13)$$

We finally discuss the probability density  $p_{\mathbf{z}}(\mathbf{x})$  for the coordinates in the normalized coherent state  $|\mathbf{z}\rangle/\sqrt{\langle \mathbf{z}|\mathbf{z}\rangle}$  such that

$$p_{\mathbf{z}}(\mathbf{x}) = \frac{|\langle \mathbf{x}|\mathbf{z}\rangle|^2}{\langle \mathbf{z}|\mathbf{z}\rangle}. \quad (6.14)$$

We recall that (6.14) is also called, especially in the context of the theory of quantum chaos the Husimi representation for the localized state on the sphere  $|\mathbf{x}\rangle$ . Let  $\mathbf{z} = \cosh|\mathbf{l}|\bar{\mathbf{x}} + i\frac{\sinh|\mathbf{l}|}{|\mathbf{l}|}\mathbf{l} \times \bar{\mathbf{x}}$  (see (2.17)), so  $\bar{\mathbf{x}}$  corresponds to the position and  $\mathbf{l}$  to the angular momentum of a particle on a sphere. From computer simulations it follows that for small enough  $|\mathbf{l}|$  the function  $p_{\mathbf{z}}(\mathbf{x})$  is peaked at  $\mathbf{x} = \bar{\mathbf{x}}$ . Therefore the parameter  $\mathbf{x}$  in the formula (2.17) can be really regarded as the classical position for a particle on a sphere.

## VII. DISCUSSION

In this work we have introduced the Bargmann representation referring to the coherent states for a particle on a sphere. The very general construction of the Bargmann space, where the configuration space is a symmetric space has been recently introduced by Stenzel [7]. As remarked by Hall [8] such construction generalizes the case discussed herein with the configuration space coinciding with the sphere  $S^2$ . Nevertheless, it does not fit into the usual scheme of construction of Bargmann spaces by means of the resolution of the identity for the corresponding coherent states. More precisely, the coherent states are not utilized at all in [7]. The approach taken up in [7] is very general and as far as we are aware the observations of our work are one of the first concrete nontrivial example of the general construction discussed in [7]. On the other hand, the construction introduced by Stenzel shows that the formalism introduced in this paper has a deeper mathematical context. We finally point out that the results obtained herein seem to be of interest also in the theory of classical orthogonal polynomials in the complex domain as well as the theory of heat kernels.



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